(10)] with another candidate moment strain

$$\bar{\kappa} = w'' / (1 + w'^2)^{3/2} \tag{11}$$

that, at first, may seem more likely to be correct. The difference in the two geometrically is that while $\bar{\kappa}$ is the curvature of the curve $w$ vs $x$ (which has no physical significance for this problem), $\kappa$ is approximately (exactly for $s' = 1$) the physical curvature of the beam elastic axis which is nearly inextensional for small strain. Thus, the bending moment is proportional to $\kappa$, and the strain energy can now be written as

$$U = \frac{1}{2} \int_0^\ell [EA(s' - 1)^2 + EI\kappa^2] \mathrm{d}x \tag{12}$$

The virtual work, including terms associated with a follower force $-Pi(\ell)$, is then given by

$$\delta U + P[\delta\xi(\ell)\cos\beta(\ell) + \delta w(\ell)\sin\beta(\ell)] = 0 \tag{13}$$

with $U$ given by Eq. (12) and $s'$, $\kappa$, $\sin\beta$, and $\cos\beta$ expressed from Eqs. (7) and (10). Boundary conditions follow naturally from Eq. (13) that will not and cannot contain quantities like $\sin w'$ as do the boundary conditions depicted in Ref. 1. Thus, it is apparent that the boundary conditions given in Ref. 1 are incorrect, strictly speaking, although numerical results may not be appreciably different when the present equations are used. It is also now apparent that an analysis similar to that of Ref. 1 can be formulated that is restricted to small strains but has no restriction on rotations other than the singularity at $\beta = 90$ deg. A dynamic analysis linearized about the equilibrium condition thus obtained would be consistent with the state of the art in elastic stability theory and may lead to improved understanding of the phenomena discussed in Ref. 1.

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# Reply by Authors to D. H. Hodges


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THE authors would like to express their thanks to Dr. Hodges for his Comment and to point out the following:

1) An exact investigation of the stability of elastic systems subjected to follower forces entails considerable computational difficulties. This is evident if the mathematical formulation of such a problem is based on small strains but large rotations. The expressions for the axial and moment strain, strain energy, $\sin\beta$, and $\cos\beta$ given by Dr. Hodges are well known.[1-3] The authors, in order to overcome these mathematical difficulties, have chosen to employ the theory of intermediate deformation[4] (small strains but moderately large rotations) which gives excellent results within a wide range.

2) Clearly, the difference between $\sin\beta$ (which theoretically should be present in the boundary conditions) and $\sin w'$ (where $w' = \tan\beta$) due to the small rotations involved is negligible. However, the numerical results presented in the work under discussion are associated with Eqs. (20), which are based on the following aprroximations: $\sin\beta \simeq w'$, $\cos\beta \simeq 1$ and $u = -w''$. These approximations are consistent with the aforementioned theory. A detailed derivation of all nonlinear stability equations of this work is presented in Ref. 5, where use of these approximations is made. The authors are presently employing a more inclusive theory to solve the same problem.

3) An extension of the foregoing work in which several aspects of this problem are clarified is available in Ref. 6.

4) From this work and that of Ref. 6 it is indicated that the nonlinear terms have the following result: the range of values of parameters for which according to linear stability analysis (either dynamic or static) flutter occurs might be appreciably reduced, if a nonlinear static analysis is employed.

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# Comment on "Potential of Transformation Methods in Optimal Design"


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IN a recent Note,[1] Belegundu and Arora presented an approach for computing the derivatives of a penalty function directly without calculating derivatives of individual constraints. The penalty function acts as an equivalent constraint replacing all other constraints. Their approach permits computation of the first derivatives of the penalty function with one forward and backward substitution (FBS) of the stiffness matrix equation. Newton's method is a powerful technique, for it finds a minimum of a quadratic

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**Table 1 Number of forward and backward substitutions (FBS) required for using second order method (when NCON > NDV)**

| Method | Displacement vector | First derivatives | Number of FBS needed in computing Hessian matrix (second derivatives): Based Ref. 2 | Approximate with error minimization | Total FBS |
|---|---|---|---|---|---|
| Refs. 1 and 2 | 1 | 1 | 2(NDV) + 1 | — | [2(NDV) + 3] |
| Refs. 3, 5-6 | 1 | NDV | — | — | NDV + 1 |

**Table 2 Number of forward and backward substitutions (FBS) required for using second order method (when NCON < NDV)**

| Method | Displacement vector | First derivatives | Number of FBS needed in computing Hessian matrix (second derivatives): Based Ref. 2 | Approximate with error minimization | Total FBS |
|---|---|---|---|---|---|
| Refs. 1 and 2 | 1 | 1 | 2(NDV) + 1 | — | [2(NDV) + 3] |
| Refs. 3, 5-6 | 1 | NCON + 1 | — | — | NCON + 1 |

function in a single step. Therefore, Belegundu and Arora suggested an approach of combining Haug's approach of computing second derivatives[2] with their computational scheme so that the advantages of a second order method such as Newton's could be retained. The idea is attractive, but the authors do not provide a clear explanation of the relative costs in using their approach as compared to other methods—methods that are also based on Newton's method (see, e.g., Ref. 3 or 4) but require only first derivatives.

The purpose herein is to carry out a comparative cost evaluation of Belegundu and Arora's approach for using Newton's method with that proposed by others, but based on approximate second derivatives.[3-6] This can be done on the basis of normal operation count in the solution of the matrix equations. If we continue to count the number of forward and backward substitutions as a measure of the overall computational cost, then the efficiency of the two methods involved can be compared easily. Let N be the number of displacement degrees of freedom, NDV the number of independent design variables, and NCON the number of active constraints. Tables 1 and 2 show the results of the matrix operation count for two situations when NCON > NDV and vice versa. In both cases the approximate Hessian-based Newton method[4] appears more attractive than Belegundu and Arora's suggested approach. The minimum difference occurs (when NCON = NDV), and then Belegundu and Arora's approach is approximately twice as costly as those based on the approximate Newton method. For other values of NCON (for example, NCON/NDV = 1/5), the penalty in computational cost of combining Refs. 1 and 2 is roughly of the order of 9:1. In addition, the one-dimensional minimization required for the approximate Newton method is a very inexpensive operation[3,4] since it can be based on explicit constraint approximations.[7]

The savings in the computational cost of first derivatives in the Belegundu and Arora approach thus has no significant effect when the overall cost is considered. In fact, it may lead to the following computational problems.

With only the derivatives of a single equivalent constraint at one's disposal, it is not clear how one would obtain explicit approximation for individual constraints, if line search is necessary. And if a Taylor series expansion for the single equivalent constraint is used, there is a genuine question concerning the accuracy and conservativeness of such approximations. If line search is not used, which of course can be done using a strict Newton method, there is still the question about how one would avoid the common oscillating phenomenon inherent in strict Newton method when applied to solve nonquadratic types of structural problems.

Finally, the concept of equivalent penalty functional[1] does not extend to those classes of penalty functions (or transformation methods) which are used in conjunction with a second order method, approximate second derivatives and error minimization schemes.[3] This is because the latter requires the derivatives of the individual constraints to minimize the error in the approximation of the second derivatives, whereas, in Ref. 1, the derivatives of the individual constraints are not available.

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